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Balanced graphs and noncovering graphs

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Abstract

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Probabilistic arguments show that triangle-free noncovering graphs are very common. Nevertheless, few specific examples are known. In this paper we describe a simple method of constructing a large family of such graphs. We first construct graphs that have very restricted diagram orientations and then show that identifying certain sets of vertices in one of these graphs produces a noncovering graph.

Introduction

A noncovering graph is a graph that cannot be oriented as the (Hasse) diagram of a partially ordered set. Probabilistic arguments show that noncovering graphs of any girth are very common (see Bollobás, Brightwell and Nešetřil [4] or Nešetřil and Rödl [7]), but nevertheless, they are elusive and few specific examples of girth greater than 3 are known. The most familiar examples are the Mycielski graphs M_n based on cycles of length n, n odd. M_5 is illustrated in Fig. 2.

In this paper we describe a simple method of constructing noncovering graphs including all the graphs M_n and many new examples. The method consists of first constructing a 'balanced graph' which has very restricted diagram orientations and then identifying certain sets of vertices produces a noncovering graph. We show that balanced graphs necessarily have girth 4. So the method does not produce noncovering graphs of large girth. However, the construction does not strictly require a balanced graph at the outset, but only a graph containing a 'balanced cycle'. So it is possible that it may be generalised to produce noncovering graphs of large girth.

In the main part of the paper we describe balanced graphs and cycles and give a formal proof of two versions of our construction of noncovering graphs. The first includes a new and very short proof of the well-known fact that the Mycielski graphs are noncovering. We conclude with a series of examples of new noncovering graphs constructed by our method.

Notation

We follow the notation of Pretzel [8] closely, but repeat the main definitions here for the convenience of the reader. All graphs considered will be assumed to be finite, connected and free of loops and multiple edges.

A *walk* W from a vertex v_1 to a vertex v_{k+1} of a graph G is a sequence of vertices $W = v_1, v_2, \ldots, v_{k+1}$ such that for each $i \in \{1, \ldots, k\}$, the pair (v_i, v_{i+1}) is an edge of G. These edges are the edges of W. The *length* k, of W is denoted by $|W|$, W is *empty* if $k = 0$. It is *simple* if all its vertices are distinct. Given two walks W_1 and W_2 such that the last vertex of W_1 is the first vertex of W_2 we can form their concatenation, $W_1 + W_2$, The *inverse* walk $-W$ is obtained by reversing the order of the vertices of W.

A *circuit* K is a walk $K = v_1, \ldots, v_{k+1}$ in which the first vertex v_1 is the same as the last v_{k+1}. We shall identify K with the shifted circuit $v_2, \ldots, v_{k+1} = v_1, v_2$, but distinguish K and the inverse circuit $-K$. A circuit is *simple* if $k > 2$ and the vertices $v_1, \ldots, v_k$ are distinct. A circuit is *trivial* if it is a sum of walks of the form $W - W$. Thus a circuit of the form $U + V - V - U + W - W$ is $[(U + V) - (U + V)] + [W - W]$ and hence trivial. Circuits are *equivalent* if they differ only by trivial circuits.

The subgraph P of G consisting of the vertices and edges of a simple walk or circuit is called a *path* or *cycle*.

A set of simple circuits $B = \{K_1, \ldots, K_n\}$ is called a *circuit basis* if every circuit is equivalent to a concatenation of elements of B and their inverses. It is easily seen that the cycles corresponding to a circuit basis form a cycle basis in the usual sense. The converse is, however, not necessarily true. Nevertheless the usual construction for a cycle basis starting with a spanning tree of the graph also yields a circuit basis.

Definition 1. Given a walk $W = v_1, \ldots, v_{k+1}$ in a graph G and an orientation R of the edges of G, we call an edge (v_i, v_{i+1}) a *forward* edge of W if R directs it towards v_{i+1}. Otherwise the edge is a *backward* edge. If W is not simple the same edge may occur several times and we distinguish each of these occurrences, so it is quite possible for the same edge to occur twice as a forward edge and ten times as a backward edge. The set of forward edges of W (with multiplicities) is denoted by W_R^+ and the set of backward edges is denoted by W_R^-. The *flow difference* of W is

$$f_R(W) = |W_R^+| - |W_R^-|.$$

It is obvious that the flow difference of the concatenation of two walks is just the sum of their individual flow differences and that a trivial circuit has zero flow difference. Hence equivalent circuits have the same flow differences and the flow differences of all circuits are completely determined by the flow differences of the elements of a circuit basis.

Diagram orientations R are characterized by the condition that $|K_R^-| \geq 2$ for all nontrivial circuits K (as $(-K)_R^- = K_R^+$, the same condition for K_R^+ follows). This condition must be verified for all simple circuits—it is not sufficient to verify it only on a circuit basis.

We now come to the central concept of this paper.

Definition 2. Let R be a diagram orientation of the graph G and K a circuit of G.

(i) K is called *R-balanced* if $f_R(K) = 0$.

(ii) R is a *balanced orientation* if all circuits are R-balanced.

(iii) K is a *balanced circuit* if it is R-balanced for all diagram orientations R.

(iv) G is a *balanced graph* if all its circuits are balanced circuits or equivalently if all its diagram orientations are balanced orientations.

We shall abuse notation and call the cycle underlying a simple balance circuit a balanced cycle. It should be emphasised that the use of any of these terms implicitly includes the statement that the graph G is a covering graph.

Of course, trivial circuits are always balanced circuits, but the usefulness of the concept lies in the fact that 4-cycles are always balanced. Further obvious facts are that balanced circuits have even length and so a graph with a balanced orientation must be bipartite. Conversely, any bipartite graph has an obvious balanced orientation.

Clearly, a graph with a circuit basis of balanced circuits is a balanced graph. It is not immediately clear that the graph is also balanced if it has a cycle basis of balanced cycles (that is shown in Bandelt, Pretzel and Rival [2]). Bandelt and Rival [3] have shown that a planar balanced graph has a cycle basis of 4-cycles, in [2] an example is given to show that this is not true in general.

We conclude this section by showing that any balanced graph that is not a tree contains 4-cycles.

Proposition 1. *A balanced graph G that is not a tree has girth* 4.

Proof. G is bipartite. So its girth is at least 4. Suppose G has girth at least 6 and let C be a cycle of G. Let R be a balanced orientation of G and let S be obtained from R by reversing a single edge $e \in C$. For every nontrivial circuit K,

$$|K_S^-| \geq |K_R^-| - 1 = |K|/2 - 1 \geq 3 - 1 = 2.$$

Hence S is a diagram orientation. On the other hand for a simple circuit K with C

as its underlying cycle,

$$f_S(K) = f_R(K) \pm 2 \neq 0.$$

Hence S is not a balanced orientation, contradicting the assumption that G is a balanced graph. □

Noncovering graphs from balanced graphs

Suppose that G is a graph containing a balanced cycle C, and that G' is obtained from G by identifying certain vertices of C. Then any orientation R' of G' induces an orientation R of G, and so the fact that C is a balanced cycle places constraints on R'. It is quite easy to ensure that these constraints cannot be satisfied by a diagram orientation. Thus the main difficulty in this approach to the construction of noncovering graphs is finding balanced cycles.

Definition. Let G be a graph. We say that a surjective graph homomorphism $f: G \to G'$ is an *identification of* G if the induced map $f: E(G) \to E(G')$ is surjective.

Note that to avoid loops in G', we do not permit identification of adjacent vertices. We shall frequently describe the identification in an intuitive manner and leave a formal definition of the map f to the reader. We will usually represent vertices and edges of G' by any of their pre-images, using several different pre-images for the same object if that is convenient. It is quite possible that an identification of a covering graph produces a noncovering graph, but the converse is not possible.

Proposition 2. *Let $f: G \to G'$ be an identification and let R' be a diagram orientation of G'. Then the orientation $R = f^{-1}(R')$ is a diagram orientation of G.*

Proof. Any nontrivial circuit K of G induces a circuit $K' = f(K)$ of G' and $|K_R^-| \geqslant |K_{R'}^{1-}| \geqslant 2$. □

Our main theorem which follows gives one way of identifying vertices in a balanced cycle that will produce a noncovering graph. This theorem has many variations and we shall prove one of them in the next section because it leads to some interesting examples. However, the basic idea is very simple and would only be obscured by an attempt at maximum generality.

Theorem. *Suppose $K = v_1, v_2, \ldots, v_{k+1} = v_1$ is a simple chord-free balanced circuit of a graph G and $p > 3$ is an odd divisor of k. Construct G' from G by identifying vertices v_i and v_j of K if $i \equiv j \pmod p$. Then G' is a noncovering graph.*

Proof. Let K' be the circuit $v_1, v_2, \ldots, v_{k+1} = v_1$ in G'. Of course K' is no longer simple. Indeed K' is the p/k – fold concatenation of the circuit

$$\bar{K} = v_1, v_2, \ldots, v_{p+1} = v_1$$

with itself. Suppose that G' has a diagram orientation R' and let R be the induced orientation of G as in Proposition 2. Then R is a diagram orientation. Hence

$$0 = f_R(K) = k/p \cdot f_{R'}(K').$$

Thus $f_{R'}(K') = 0$. But K' is an odd cycle and hence it cannot have zero flow difference. Thus the assumption that G' has a diagram orientation leads to a contradiction. □

It is quite possible that G' contains a triangle and so is trivially a noncovering graph. So care has to be exercised in choosing the vertices to be identified to prevent that happening. That is the reason for excluding the case $p = 3$, even though the proof works just as well then.

We illustrate our theorem by producing an easy construction of M_5 that immediately shows that it is a noncovering graph. As in all our examples we produce balanced cycles by beginning with a balanced graph with a cycle basis of 4-cycles. Recall that the face cycles of a planar graph form a cycle basis (see e.g. Giblin [6, p. 38]). We recall that a face cycle is one that has no vertices of the graph in its interior.

Example 1. Let G_1 be the graph illustrated in Fig. 1. The graph G_1 is balanced, since all its face cycles are 4-cycles. If we apply Theorem 1 to the boundary circuit

$$K_1 = u_1, u_2, \ldots, u_{11} = u_1$$

of the outer region of G_1 with $p = 5$, we find $G_1' = M_5$. This is shown in Fig. 2. Fig. 3 illustrates how this construction extends to M_n for arbitrary odd n.

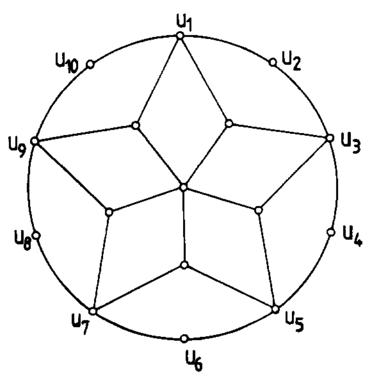

Fig. 1.

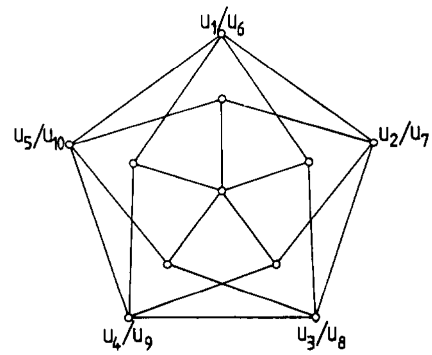

Fig. 2.

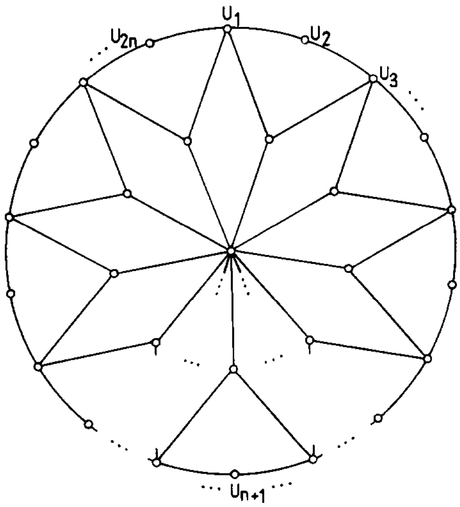

Fig. 3.

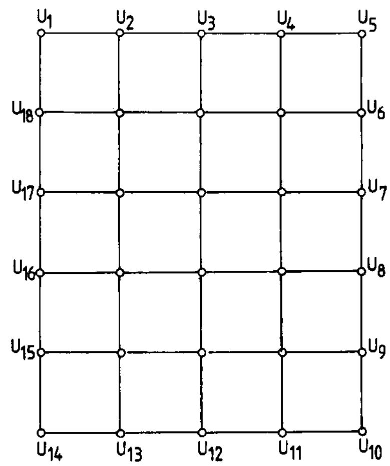

Fig. 4.

A generalization and some further examples

The identification of the vertices of the boundary cycle of a planar graph is strongly reminiscent of the way closed surfaces are constructed from a polygon in topology. There however, some edges may be reversed before they are identified. If we emulate this in our construction we obtain more complicated conditions but we may still be able to show that they cannot be satisfied by a diagram orientation. Before formulating a (fairly) general version of our theorem, we give an example to show that the generalization really is worthwhile.

Example 2. Let G_2 be the graph illustrated in Fig. 4. Then G_2 is balanced for the same reasons as G_1. We again choose the boundary circuit $K_2 = u_1, \ldots, u_{19} = u_1$ as our balanced circuit. Now however we identify the top and bottom paths directly while we reverse the left path before identifying it with the right hand one. We can represent this by the scheme used in topology (see Fig. 5). The graph G_2' obtained by this identification is illustrated in Fig. 6. The image of K_2 in G_2' has the form

$$K' = X + Y - X + Y,$$

where X is a 4-circuit and Y is a 5-circuit. Hence if G_2' has a diagram orientation R' and R is the orientation of G_2 inducing R' we have

$$0 = f_R(K) = f_{R'}(K') = 2f_{R'}(Y) \neq 0.$$

This contradiction shows that G_2' is a noncovering graph.

Note that G_2' is 4-regular. So it is a new example of a 4-regular 4-chromatic graph as well as being an example of a 4-regular noncovering graph. It is well

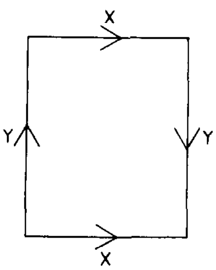

Fig. 5.

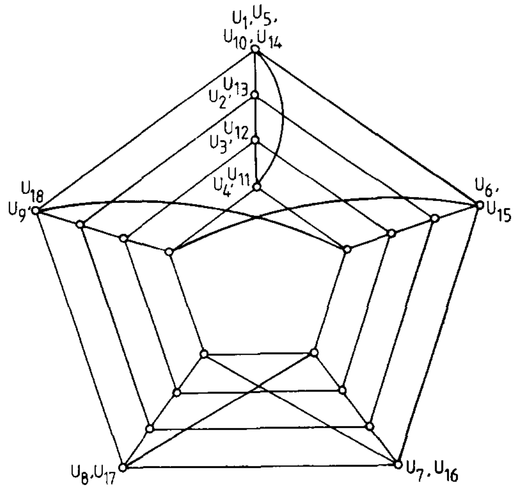

Fig. 6.

known that if the girth of a graph G is greater than its chromatic number, then G is a covering graph, see Aigner and Prins [1]. The Chvatal 4-regular, 4-chromatic graph (see Bondy and Murty [5, p. 241]) is also a noncovering graph, however in that example two edges can be removed leaving a noncovering graph. Like the Mycielski graphs, our example is edge critical.

In order to simplify the statement of the theorem we make the following definition.

Definition 3. Let $U = u_1, u_2, \ldots, u_k$ and $V = v_1, v_2, \ldots, v_k$ be two walks in a graph G. We will say we are identifying U and V if we identify u_i with v_i for all $i = 1, \ldots, k$, and that we are identifying U with $-V$ if we are identifying u_i with v_{k-i} for all $i = 1, \ldots, k$.

Theorem 2. *Let G be a graph with a simple, chord-free balanced circuit $K = v_1, v_2, \ldots, v_{k+1} = v_1$. Suppose further that $n_i, a_i, b_i, i = 1, \ldots, l$ are nonnegative integers such that:*

(i) $|K| = \sum_{i=1}^{l} n_i(a_i + b_i)$.

(ii) *The equation* $0 = \sum_{i=1}^{l} f_i(a_i - b_i)$ *has no solutions with* $|f_i| \leq n_i$ *and* $f_i \equiv n_i \pmod 2$.

Write K as the concatenation (in any order) of simple walks W_i^j of length n_i, where i runs from 1 to l and j runs from 1 to $a_i + b_i$. (This is possible by condition (i).) *Let the graph G' be obtained by identifying*

$$W_i^1, \ldots, W_i^{a_i}, -W_i^{a_i+1}, \ldots, -W_i^{a_i+b_i} \quad \textit{for all } i = 1 \textit{ to } l.$$

Then G' is a noncovering graph.

Proof. Suppose that R' is a diagram orientation of G'. Then for the induced walk $W' = W_i^1$ in G', $f_i = f_{R'}(W') \equiv |W'| = n_i \pmod 2$. Now let R be the diagram orientation of G obtained by giving each edge the orientation of the induced edge in G' (see Proposition 2). Then for all $i = 1, \dots, l$,

$$f_R(W_i^j) = f_i \quad \text{for } j = 1, \dots, a_i$$

and

$$f_R(W_i^j) = -f_i \quad \text{for } j = a_i + 1, \dots, a_i + b_i.$$

Hence

$$0 = f_R(K) = \sum_{i=1}^{l} \sum_{j=1}^{a_i+b_i} f_R(W_i^j) = \sum_{i=1}^{l} f_i(a_i - b_i),$$

contradicting (ii). □

Example 2 corresponds to the case $|K| = 18$, $a_1 = b_1 = 1$, $a_2 = 2$, $b_2 = 0$, $n_1 = 4$ and $n_2 = 5$ in Theorem 2. Clearly $2f_2 = 0$ cannot be satisfied for odd f_2. The example therefore leads to a whole family of noncovering graphs using the same identification, where the width of the initial grid may be any number greater than 3 and its height may be any odd number greater than 4.

If we ensure that the walks W_i^j in Theorem 2 reduce to circuits in G' then we can restrict the values f_i in condition (ii) further, because the flow difference of a circuit K in a diagram orientation R satisfies $|f_R(K)| \leq \max(0, |K| - 4)$. We state this formally in a corollary and then demonstrate its use in a final example.

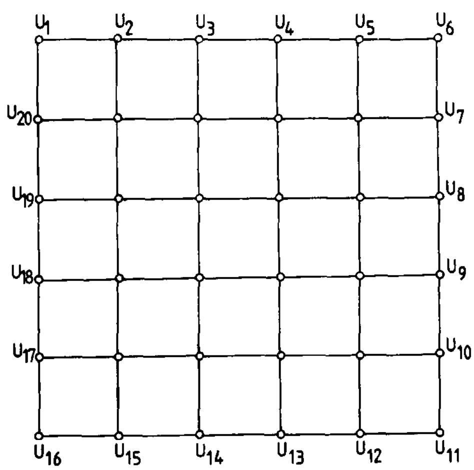

Fig. 7.

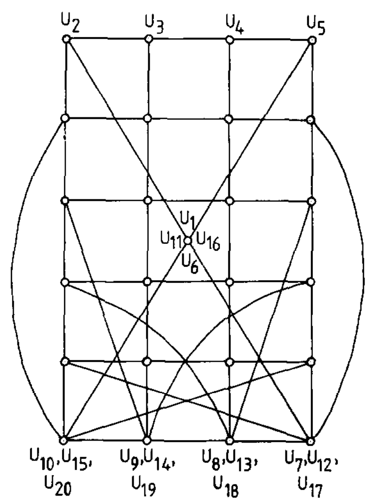

Fig. 8.

Corollary. *Suppose that for $i = 1, \ldots, k$ in Theorem 2, W_i^j reduces to a circuit in G' then condition* (ii) *may be replaced by*:

(ii)′ $0 = \sum_{i=1}^{l} f_i(a_i - b_i)$ *has no solutions with*

$$|f_i| \leqslant \max(0, n_i - 4) \quad \textit{for } i = 1, \ldots, l,$$

$$|f_i| \leqslant n_i \quad \textit{for } i = k+1, \ldots, l,$$

$$f_i \equiv n_i \pmod 2 \quad \textit{for } i = 1, \ldots, l.$$

Example 3. Let G_3 be the balanced graph illustrated in Fig. 7 and let $K = v_1, v_2, \ldots, v_{21} = v_1$ be the boundary circuit of its outer region. Construct G_3' by identifying v_i with v_{i+s} for $i = 6, \ldots, 16$. That corresponds to a choice of $a_1 = 1$, $a_2 = 3$, $b_1 = b_2 = 0$, $n_1 = n_2 = 5$ in Theorem 2 (see Fig. 8). Then the only choices of odd f_1 and f_2 with $|f_i| \leqslant 5$ such that $f_1 + 3f_2 = 0$, are $f_1 = \pm 3$ and $f_2 = \mp 1$. But as the walk W_1^1 reduces to a 5-circuit in G', f_1 must satisfy $f_1 \leqslant 1$. Hence the restricted form of condition (ii) is satisfied. So G_3' is a noncovering graph.

References

[1] M. Aigner and G. Prins, k-orientable graphs, Mathematics Department, Free University Berlin, preprint, 1980.

[2] H.-J. Bandelt, O. Pretzel and I. Rival, manuscript in preparation, 1988.

[3] H.-J. Bandelt and I. Rival, Diagrams, orientations and varieties, Department of Computer Science, University of Ottawa, preprint, 1988.

[4] B. Bollobás, G. Brightwell and J. Nešetřil, Random graphs and covering graphs of posets, Order 3 (1986) 245–257.

[5] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976) 241.

[6] P.J. Giblin, Graphs, Surfaces and Homology (Chapman and Hall, London, 1977) 38.

[7] J. Nešetřil and V. Rödl, On a probabilistic graph theoretical method, Proc. Amer. Math. Soc. 72 (1978) 417–421.

[8] O. Pretzel, On graphs that can be oriented as diagrams of ordered sets, Order 2 (1985) 25–40.